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Rabindra N. Bhattacharyya  
*Utah State University*

Donald L. Snyder  
*Utah State University*

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Rabindra N. Bhattacharyya

and

Donald L. Snyder

Department of Economics  
Utah State University  
Logan, Utah 84322

# STUMPAGE PRICE UNCERTAINTY AND THE OPTIMAL ROTATION OF A MULTIPLE USE FOREST: AN APPLICATION OF SANDMO MODEL

The Faustmann model has played a key role in the determination of optimal forest rotations. Faustmann (1849) developed a simple and deterministic competitive economic model, the objective of which was to maximize the present value of perpetual returns to the fixed factor, a unit of timber land. The optimal rotation problem thus viewed is a timber management problem abstracting from the multiple use characteristics of a forest stand and any environment of uncertainty. Hartman (1976) developed a modified deterministic Faustmann model where a standing forest has value in the form of "recreation", a general term used to capture non-timber forest uses. He did not consider regeneration costs and the costs of making recreation available to users.

This paper considers an alternative model formulation that includes the net values of a multiple use forest operated under stumpage price uncertainty and forest owners with risk aversion. By use of the theory of competitive firm under price uncertainty developed by Sandmo (1971), a more generalized Faustmann rule under conditions of uncertainty is developed.

## Optimal Rotation Age Under Price Uncertainty

In the analysis presented the forest consists of a single homogeneous tree population distributed uniformly and grown on an initially barren land. The forest manager is assumed to be operating in a perfectly competitive market and to have perfect knowledge of the level of the tree population, the demand for recreation, and the costs associated

with providing recreation and the regeneration and harvesting costs. That is, tree stock and net value of the flow of recreational services are assumed to be deterministic.

Let  $G(t)$  denote the stumpage value (net of harvesting cost) in a forest of age  $t$ . The value of the flow of services of the standing forest at age  $t$  (e.g. wildlife habitat, flood control, viewing, and hunting), will be referred to as  $F(t)$ , the value of "recreational" services.

Consider a forest stand consisting of a stock of homogeneous trees planted and used along with other cooperating factors (such as inputs for road development and maintenance, campground preparation and clean-up, wildlife habitat improvement programs, preserving the stock of trees) for producing a flow of recreational service,  $Q$ . Over time,  $Q$  is made available in a competitive market. Let  $q(t)$  denote the flow of  $Q$  at a point in time. The corresponding value of recreational service flow is  $F(t)$ . The forest stand is regenerated in an initially barren land at time  $t = 0$ , at a fixed regeneration cost,  $C_0^R$ . The input cost flow to produce and make recreational services accessible to prospective users,  $C^I(t)$ , is a function of  $q(t)$ . The maintenance cost flow for the tree stock and other durable cooperating inputs,  $C^M(t)$ , is a function of both the flow of services and of the age of the forest (assuming that ages of other durable inputs are linearly related to age of the forest). Consequently,

$$C(t) = C^I(t) + C^M(q(t), t) = C(q(t), t), \quad (1)$$

which may be called the variable cost function.

The quasi-rent function, which is value of recreational services ( $q(t)$  times exogeneously determined price of recreational services) minus variable costs is then



$$R(q(t), t) = F(q(t), t) - C(q(t), t) = R(t). \quad (2)$$

It is assumed that the forest manager considers only the stumpage price  $p$  stochastic with a subjective probability density function  $\phi(p)$  and an expected price,  $E[p] = \bar{p}$ .  $E$  is the expectation operator. Furthermore, it is assumed that planting decision when the production process starts, must be taken ex ante, i.e., before the stumpage price is known, and only on the basis of the knowledge of the price summarized in the density function. To facilitate comparison with the deterministic model, the stochastic price can be subsumed in stochastic stumpage value. If  $G(t)$  is the stumpage value (net of harvesting cost) of a forest of age  $t$  with a stochastic price, then  $G(t)$  is stochastic with a subjective density function  $f[G(t)]$  and an expected stumpage value  $E[G(t)] = \bar{G}(t)$ .

Given this, the forest manager is faced with the problem of choosing a rotation cycle that will maximize the expected net returns that can be made from maintaining a standing forest and by harvesting it. The forest manager is assumed to maximize the expected utility of discounted value of all net returns from the forest resource calculated over the infinite chain of renewal cycles. The net return from a single rotation is given by

$$V_1(t) = \int_0^T R(t)C^{-rt} dt + G(T)e^{-rT} - C_0^R, \quad (3)$$

where  $r > 0$  is the discount rate and  $G(T)$  is a random variable of stumpage value.

Given that all rotations are alike, the net return from all future rotations is given by

$$V = \frac{1}{1-e^{-rT}} \left[ \int_0^T R(t)e^{-rt} dt + G(T)e^{-rT} - C_0^R \right]. \quad (4)$$

The approach adopted here is to describe the rotation problem in terms of the classic Von Neumann-Morgenstern theory of individual decision making under uncertainty. Uncertainty in stumpage price results in a  $V$  that is stochastic. Hence, the manager must select the best of the available probability distributions for  $V$ , which are called random prospects. If we assume that the manager's behavior in solving this problem conforms to the Von Neumann-Morgenstern axioms<sup>1</sup>, then it can be inferred that the preference ordering for various random prospects can be represented by a utility function  $U[V(t)]$  and that the best prospect is found by maximizing the expected value of utility.

For a forest manager with a planning horizon running through one harvest cycle from the time  $t=0$  through  $t=T$ , the objective function to be maximized with respect to  $T$  can be written as

$$W_1(T) = E \{ U[V_1(T)] \}. \quad (5)$$

When the planning horizon is extended to an infinite sequence of identical harvest cycles the objective function to be maximized turns out to be

$$W(T) = E \{ U[V_1(T)/1-e^{-rT}] \}. \quad (6)$$

The forest manager's attitude towards risk in resource return is represented by the form of the  $U[V(T)]$ . Strict concavity in the utility function implies risk aversion. The choice of the particular form is based on its risk characteristics in terms of the measures of risk aversion developed by Arrow (1971) and Pratt (1964). In the analysis

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<sup>1</sup> See, for example, Henderson and Quandt, pp. 53-54.

here, utility is represented by a concave, continuous, and twice differentiable function of discounted net returns,  $U[V(T)]$ , where

$$U'[V(T)] > 0, U''[V(T)] < 0, \quad (7)$$

so that the forest manager is assumed to be risk averse.

For clarity and convenience of exposition, the analysis runs in terms of two cases: the Fisherian one-cycle case and the Faustmann many-cycle case.

#### Fisherian one-cycle solution

For a one-cycle time horizon, the expected utility of the discounted net return from a forest of age  $T$  is:

$$E \{ U[V_1(T)] \} = \int_0^T U \left[ \int_0^T e^{-rt} R(t) dt + e^{-rT} G(T) - C_0^R \right] f[G(T)] dG(T) \quad (8)$$

where the first integration is over the range of  $G(T)$ . Alternatively stated

$$E \{ U[V_1(T)] \} = E \left\{ U \left[ \int_0^T e^{-rt} R(t) dt + e^{-rT} G(T) - C_0^R \right] \right\}. \quad (9)$$

Differentiating (9) with respect to  $T$ , the necessary condition for an optimum is

$$E \{ U'[V_1(T)] [e^{-rT} R(T) - re^{-rT} G(T) + e^{-rT} G'(T)] \} = 0. \quad (10)$$

This implies that

$$E \{ U'[V_1(T)] [R(T) - rG(T) + G'(T)] \} = 0. \quad (11)$$

The sufficient condition for an optimum is

$$D = E \{ U''[V_1(T)] [R(T) - rG(T) + G'(T)]^2 e^{-rT} + U'[V_1(T)] [R'(T) - rG'(T) + G''(T)] - r[R(T) - rG(T) + G'(T)] \} < 0. \quad (12)$$



If  $[R'(T) - rG'(T) + G''(T)] < r [R(T) - rG(T) + G'(T)]$ , then  $D < 0$  is satisfied.

It is assumed that (11) and (12) determine a nonzero, finite and unique solution  $T$ , say  $\hat{T}$  to the present maximization problem. Under certainty, the solution  $T$  is characterized by the equality between net gain from marginal time and opportunity cost of marginal time. To allow for the comparison between the competitive optimal rotation under conditions of certainty and uncertainty, following Sandmo (1971) the problem is posed as follows: What is the optimal rotation time under uncertainty compared to the situation where the stumpage price is known to be equal to the expected value of the original distribution. The latter time is referred to as the deterministic time.

Now the first order condition (11) can be rewritten as

$$E \{ U'[V_1(T)] R(T) \} + E \{ U'[V_1(T)] G'(T) \} = E \{ U'[V_1(T)] rG(T) \}. \quad (13)$$

Subtracting  $E\{U'[V_1(T)]E[rG(T)]\}$  from and adding  $E \{ U'[V_1(T)] E[G'(T)] \}$  to both sides of (13) and remembering that  $E[rG(T)] = r\bar{G}(T)$  and  $E[G'(T)] = \bar{G}'(T)$ ,

$$E \{ U'[V_1(T)] [R(T) + \bar{G}'(T) - r\bar{G}(T)] \} = E \{ U'[V_1(T)] [rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)] \}. \quad (14)$$

Since  $E[V_1(T)] = \int_0^T R(t)e^{-rt}dt + E[G(T)]e^{-rT} - C_0^R$  (from the definition of  $V_1(T)$ ), we have  $V_1(T) = E[V_1(T)] + [G(T) - \bar{G}(T)]e^{-rT}$ . Given the concavity of  $U$ , it then follows that

$$U'[V_1(T)] < U' \{ E[V_1(T)] \} \quad \text{if } G(T) > \bar{G}(T). \quad (15)$$

Then,

$$U'[V_1(T)] [rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)] < U'\{E[V_1(T)]\} [rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]. \quad (16)$$

This inequality holds for all  $G$  and  $G'$ .<sup>2</sup> Taking expectations on both sides of (16) and noting that  $U'\{E[V_1(T)]\}$  is a given number,

$$E\{U'[V_1(T)] [rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]\} < U'\{E[V_1(T)]\} E[rG(T) - r\bar{G}(T) + \bar{G}'(T) - G'(T)]. \quad (17)$$

But here the right-hand side is equal to zero by definition, and therefore the left-hand side is negative. Consequently, the left-hand side of (14) is also negative. This can be written as

$$E\{U'[V_1(T)]\} [R(T) + \bar{G}'(T) - r\bar{G}(T)] < 0. \quad (18)$$

Since marginal utility is positive, this implies that

$$R(T) + \bar{G}'(T) < r\bar{G}(T). \quad (19)$$

Inequality (19) shows that the expected utility maximizing rotation time  $T$  is characterized by the expected net return of marginal time,  $R(T) + \bar{G}'(T)$ , being less than the expected opportunity cost of marginal time  $r\bar{G}(T)$ .

This implies that under stumpage price uncertainty, optimal rotation length is longer than the deterministic optimal rotation length characterized by,  $R(T) + G'(T) = rG(T)$ , where the deterministic stumpage price/value is equal to the expected price  $\bar{p}$  /value  $\bar{G}$ . This result is supported by the finding of Norstrom (1975) and may be due to the

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<sup>2</sup> See Sandmo

entrepreneur's desire for greater availability of inventory to meet uncertain future price.

Inequality (19) may be called a generalized one-cycle Fisherian rule under price uncertainty. The absence of a net benefit from recreational service (i.e., when  $R(T) = 0$ ), turns (19) to the simple Fisherian solution  $\bar{G}'(T)/\bar{G}(T) < r$ , under price uncertainty and implies a longer rotation length than under certainty (where the rule is,  $G'(T)/G(T) = r$ ) but shorter than the generalized solution characterized by (19).

#### Faustmann many-cycle solution

Here the objective function to be maximized is (6) and the necessary condition for an optimum is

$$E \{ U'[V_1(T)] [R(T) + G'(T) - \frac{1}{\lambda} (\int_0^T R(t)e^{-rt} dt + G(T) - C_0^R)] \} = 0, \quad (20)$$

where  $\lambda = (1 - e^{-rT})/r$ .

Using the same procedure as followed for the one-cycle case (with some additional terms), it can be shown that

$$E \{ U'[V_1(T)] [R(T) + \bar{G}'(T) - \frac{1}{\lambda} (\int_0^T R(t)e^{-rt} dt + \bar{G}(T) - C_0^R)] \} < 0. \quad (21)$$

Given positive marginal utility, this implies that

$$R(T) + \bar{G}'(T) < \frac{1}{\lambda} (\int_0^T R(t)e^{-rt} dt + \bar{G}(T) - C_0^R). \quad (22)$$

Inequality (22) can then be called the generalized Faustmann rotation rule under stumpage price uncertainty. It indicates again that under stumpage price uncertainty the optimal rotation length is longer

than the deterministic rotation length characterized by,  $R(T) + G'(T) = \frac{1}{\lambda} \left( \int_0^T R(t) e^{-rt} dt + G(T) - C_0^R \right)$ , where the deterministic stumpage value is equal to the expected value  $\bar{G}$ . Again, under conditions of certainty as well as under conditions of uncertainty, the Faustmann many-cycle rule implies a shorter rotation period than the Fisherian one-cycle rotation period. This occurs because the effective interest rate gets inflated in the former case.

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